① chapter 4. Stability of Stochastic Implusive System with Time Delay.

$$\begin{cases} dx(t) = f(t, x_t)\,dt + g(t, x_t)\,dW(t). & t = \tau_k. \quad (4.1a) \\ \Delta x(t) = F(t, x_{t^-}). & t = \tau_k \quad (4.1b) \\ x_{t_0} = \phi(s), \quad s \in [-r, 0] & \quad (4.1c). \end{cases}$$

Main interest: (1) mean square (m.s) global asymptotic stability

(2) problem of stabilization by implusive controller.

Similar to ODE. 4.1 Stability analysis by Lyapunov technique.

4.2 Stabiliry analysis by comparison method.

Assumption. A1:

$\exists\, 0 \le \rho_1 \le \rho$, $\forall \tau_k \in \mathbb{R}_+$ $x \in PF([-r, 0]; D)$, $D \subset \mathbb{R}^n$, If $E[\|x_{\tau^-}\|^2] \le \rho_1$ $\Rightarrow E[\|x(\tau_k)\|^2] < \rho$. | solution be bounded after any implulsive effect as long as it is bounded befor implulsive moment.

Assumption A2: $\forall k$. $T_{sup} := \sup\{\tau_k - \tau_{k-1}\} < \infty$ $T_{inf} := \inf\{\tau_k - \tau_{k-1}\} > 0$.

4.1. Lyapunov technique. refer to page 8.

Theorem 4.1. $\exists\, a, c \in \underset{\text{concave}}{\underline{k_c}}$, $b \in \underset{\text{convex}}{\underline{k_v}}$. $p \in PF\,C(\mathbb{R}_+^-, \mathbb{R}_+)$. Let $V \in \mathcal{V}^{1,2}$ $C([-r, \infty) \times \mathbb{R}^n, \mathbb{R}_+)$. s.t:

(i) $\forall (t, \psi(0)) \in [r, \infty) \times S(\rho)$.

$$\underset{\text{convex lower bound}}{b(\|\psi(0)\|^2)} \le V(t, \psi(0)) \le \underset{\text{concave upper bound}}{a(\|\psi(0)\|^2)}.$$

(ii) $\forall t \ne \tau_k \in \mathbb{R}_+$, $\psi \in PF\,C([-r, \infty) : S(\rho))$.

$f(x) \le c f(x)$
$f(x) \le -c f(x)$

$$L V(t, \psi) \le p(t)\, C\,V(t, \psi(0)) \text{ (a.s).} \quad \text{unstability}$$

provided that $\bar{g}(V(t+s, \psi(s)) \le V(t, \psi(0))$ for some $s \in [-r, 0]$. $\bar{g} \in K_3$.

(iii) at any implusive moment. $\tau_k \in \mathcal{T}$, $\psi \in PF\,C([-r, \infty); S(\rho))$.

stability condition with impulsive system. $V(\tau_k, \psi(0) + F(\tau_k, \psi(\tau_k^-)) \le \bar{g}(V(\tau_k^-, \psi(0))$ (a.s.).

with $\psi(0^-) = \psi(0)$, where $(\tau_k^-, \psi(\tau_k^-)) \in \mathbb{R}_+ \times P\mathcal{F}\,C([-r, 0] : S(\rho_1))$;

(iv) $M_1 = \sup_{t \ge 0} \int_t^{t+\tau} p(s) < \infty$. $\tau = \sup_{k \in \mathbb{N}} \{\tau_k - \tau_{k-1}\} < \infty$. $M_2 = \inf_{q > 0} \int_{\bar{g}(q)}^{q} \frac{ds}{c(s)} > M_1$.

Then, the trivial solution $x \equiv 0$ of (4.1) is uniformly asymptotically stable in the m.s. (iv) means impulsive change unstable ⇒ stable.

② proof. From condition (i), $\exists\ \hat{b}\in K_V,\ \hat{a}\in K_C.\ \ \hat{b}(s)\le b(s)\le a(s)\le \hat{a}(s).\ \forall s\in[0,\rho]$

This implies $\hat{b}(\|\psi(0)\|^2)\le V(t,\psi(0))\le \hat{a}(\|\psi(0)\|^2)$ (a.s.) ($\psi\in P\mathcal{F}([-r,0];S(\rho))$)

We split the proof into two steps:

Step A. We first prove uniform stability in the M.S., i.e., $\forall \varepsilon>0.\ \forall t_0;\ \exists \delta>0.$

s.t. $\quad E[\|\phi\|_r^2]\le\delta \Rightarrow E[\|x(t)\|^2]<\varepsilon.\quad \forall t\ge t_0.$

Step B: $\quad x\equiv 0$ is uniformly asymptotically stable in the m.s.

From Step A, we obtain that.

$\forall\gamma>0,\ \exists\eta>0.$ s.t. $E[\|\phi\|_r^2]\le\eta \Rightarrow E[\|x(t)\|^2]\le\gamma.$

We only need to find some T, $\eta>0$. s.t. $\forall\gamma>0,\ \exists T(\gamma,\eta)>0.$

$$E[\|\phi\|_r^2]\le\eta \Rightarrow E[\|x(t)\|^2]\le\gamma.\quad \forall t\ge t_0+T.$$

• proof of Step A: Take $0<\varepsilon<\rho_1$, and $x(t)$ be a solution of (4.1) in $[t_0, t_0+\beta)$.
Choose $\delta<\hat{a}^{-1}(\bar{g}(b(\varepsilon)))\Rightarrow 0<\delta<\varepsilon$ (since $0<\bar{g}(s)<s$).

If claim was not true, $\exists\ t\in[t_0,t_0+\beta)$ such that. $\quad E[\|x(t)\|^2]>\varepsilon.$

Then, define $\hat{t}=\inf\{t\in[t_0,t_0+\beta)\mid E[\|x(t)\|^2]>\varepsilon\}$.

clearly $E[\|x(t)\|^2]\le E[\|\phi\|_r^2]\le\delta<\varepsilon.\quad \forall t\in[t_0-r,t_0] \Rightarrow \underline{\hat{t}\in(t_0,t_0+\beta)}$.

and $\underline{E[\|x(t)\|^2]\le\varepsilon\le\rho_1,\ \forall t\in[t_0-r,\hat{t})}$.

For $\hat{t}$, ① $\hat{t}\ne\tau_k \Rightarrow E(\|x(\hat{t})\|^2)=\varepsilon$ (by continuity of $x(t)$).

② $\hat{t}=\tau_k$ (for some k) $\xRightarrow{\text{(maybe)}} E(\|x(\hat{t})\|^2)>\varepsilon.$, $E(\|x(\hat{t})\|)\le\rho$ (by (A1))

From Itô formula, we have.

$$V(t,x(t))=V(s,x(s))+\int_s^t LV(u,x(u))\,du+\underbrace{\int_s^t V_x g\,dW}_{E(\ \ )=0}.$$

Take the expectation, we get.

$$m(t)=E(V(t,x(t)))\le E(V(s,x(s)))+E\int_s^t LV\,du.$$
$$\le m(s)+E\int_s^t p(u)\,c(V(u,x(u)))\,du.\quad \text{(by (ii))}$$

This implies

$$D^+m(t)\le p(t)\,E\,c(V(t))\le p(t)\,m(t).\quad \forall t\ne\tau_k\in[t_0,\hat{t}]$$

(by concave of c).

provided $m(t)\ge g(\|m(t)\|_r)$. (*)

$\forall\tau_k\in(t_0,\hat{t}]$, by (iii), we obtain $m(\tau_k)\le\bar{g}(m(\tau_k^-))$.

Let $t^*=\inf\{t\in[t_0,\hat{t}]\mid m(t)\ge\hat{b}(\varepsilon)\}$. $\quad$ * $t^*>t_0$ $\quad$ * $t^*\le\hat{t}$ $\quad$ ($0<g(u)<u$)

* ($t^*>t_0$) $\quad m(t_0)\le\hat{a}(E\|\phi\|_r^2)\le\hat{a}(\delta)<\hat{a}(\hat{a}^{-1}(\bar{g}(\hat{b}(\varepsilon))))\le\bar{g}(\hat{b}(\varepsilon))<\hat{b}(\varepsilon)$.

* ($t^*\le\hat{t}$) $\quad \hat{b}(E(\|x(\hat{t})\|^2))\le m(\hat{t}). \quad\Rightarrow\quad t^*\in(t_0,\hat{t}]$

③ Next. we claim $\forall T_k \in (t_0, \bar{t}]$. $t^* \neq T_k$ (by (iii)) - $(0<\bar{g}(q)<q)$.

proof of claim: If $t^* = T_k$. then $0 \le \hat{b}(\varepsilon) = m(t^*) \le \bar{g}(m(t^{*-})) < m(t^{*-}) \le \hat{b}(\varepsilon)$),

which is impossible. we have $m(t) < \hat{b}(\varepsilon)$, $\forall t \in [t_0 - r, t^*)$ ⊛

Now, we come back to prove Step A:

• (no impulsive) $(T_{k-1} \le t_0 < t^* < T_k)$ by ($*$ $t^* > t_0$)

$m(t_0) < \bar{g}(\hat{b}(\varepsilon))$.

Let $\bar{t} = \sup\{t \in [t_0, t^*] \mid m(t) \le \bar{g}(\hat{b}(\varepsilon))\}$. $m(t^*) = \hat{b}(\varepsilon) > \bar{g}(\hat{b}(\varepsilon))$.

+ m continuous $\Rightarrow$ $\bar{t} \in (t_0, t^*)$. $m(\bar{t}) = \bar{g}(\hat{b}(\varepsilon))$. and $m(t) \ge \bar{g}(\hat{b}(\varepsilon))$.

$\forall t \in [\bar{t}, t^*]$. $\Rightarrow$ $\forall s \in [-r, 0)$ $\bar{g}(m(t+s)) \le \bar{g}(\hat{b}(\varepsilon)) \le m(t)$. ⊛

$\Rightarrow$ $D^+ m(t) \le p(t)\, C(m(t))$, $\forall t \in [\bar{t}, t^*]$.

① $\displaystyle\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{C(s)} \le \int_{\bar{t}}^{t^*} p(s)\,ds \le \int_{\bar{t}}^{\bar{t}+\tau} p(s)\,ds \le \sup \int_{\bar{t}}^{\bar{t}+\tau} p(s)\,ds = M_1$.

② $\displaystyle\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{C(s)} = \int_{\bar{g}(\hat{b}(\varepsilon))}^{\hat{b}(\varepsilon)} \frac{ds}{C(s)} \ge M_2$. (by (iv)) ③ $M_2 > M_1$

contradiction.

• (impulsive $(T_k < t^* < T_{k+1}$, $k \ge 1)$. $(t_0 \in [T_{k-1}, T_k])$.

$m(T_k^-) \le \hat{b}(\varepsilon) \Rightarrow m(T_k) \overset{\text{by (iii)}}{\le} \bar{g}(m(T_k^-)) \le \bar{g}(\hat{b}(\varepsilon))$. ($\bar{g}$ is non decreasing)

Define $\bar{t} = \sup\{t \in [T_k, t^*] \mid m(t) \le \bar{g}(\hat{b}(\varepsilon))\}$. we use the

Similar argument to get. $\bar{t} \in [T_k, t^*)$

$m(\bar{t}) = \bar{g}(\hat{b}(\varepsilon))$ $m(t) \ge \bar{g}(\hat{b}(\varepsilon))$, $\forall t \in [\bar{t}, t^*)$.

Again $D^+ m(t) \le p(t)\, C(m(t))$ $\forall t \in [\bar{t}, t^*]$

$\displaystyle\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{C(s)} \le \int_{\bar{t}}^{t^*} p(s)\,ds \le M_1$

$\displaystyle\int_{m(\bar{t})}^{m(t^*)} \frac{ds}{C(s)} = \int_{\bar{g}(\hat{b}(\varepsilon))}^{\hat{b}(\varepsilon)} \frac{ds}{C(s)} \ge M_2$ + $M_2 > M_1$, contradiction.

proof A is completed.

• Step B: From step A, we know. $\exists\, \eta > 0$. s.t. $E[\|\phi\|_r^2] \le \eta \Rightarrow E[\|x(t)\|^2] \le \rho_1$.

By (i) $\Rightarrow E(V) \le \hat{a}(E(\|x(t)\|^2)) \le \hat{a}(\rho_1)$ $\forall t \ge t_0 - r$.

Let $0 < \gamma < \rho_1$, and define $0 < M = M(\gamma) = \sup\left\{ \frac{1}{C(s)} \;\; s \in [\bar{g}(\hat{b}(\gamma)), \hat{a}(\rho_1)] \right\}$.

For q satisfy $\hat{b}(\gamma) \le q \le \hat{a}(\rho_1)$, we have $\bar{g}(\hat{b}(\gamma)) \le \bar{g}(q) \le q \le \hat{a}(\rho_1)$.

$\Rightarrow$ $M_2 \le \displaystyle\int_{\bar{g}(q)}^{q} \frac{ds}{C(s)} \le M[q - \bar{g}(q)] \Rightarrow \bar{g}(q) \le q - \frac{M_2}{M} \le q - d$.

where $d < \left(\frac{M_2 - M_1}{M}\right) < \frac{M_2}{M}$. $\forall\, \rho_1 \ge \gamma > 0$. , define

(1): Th 4.2. The stabilities ... [illegible]   6)

(4) $N = N(\gamma) = \min\{N \in \mathbb{N} \mid \hat{a}(\rho_1) < \hat{b}(\gamma) + Nd\}$.

$\exists T(\gamma) = \tau + (\cancel{r} + \tau)(N-1)$.

Next, we claim, $\exists \eta > 0$. s.t. $\forall \gamma > 0$.

$E[\|x(t)\|^2] \le \gamma$, $\forall t \ge t_0 + T(\gamma)$, $t_0 \in [\tau_{i-1}, \tau_i)$

Given $0 \le A \le \hat{a}(\rho_1)$, we use a similar argument in the proof of step A to show   6

① if $m(t) \le A$, $\forall t \in [\tau_j - r, \tau_j)$, then $m(t) \le A$. $\forall t \ge \tau_j$;

② if, in addition, $\hat{b}(\gamma) < A$, then $m(t) \le A - d$, $\forall t \ge \tau_j$. $(\tau \le r)$.

Now, we define $k(i)$ $i = 1, 2, \dots N$, s.t. $\tau_{k(i)-1} < \tau_{k(i-1)} + r \le \tau_{k(i)}$

i.e. $\begin{cases} \tau_{k(i)} - \tau_{k(i-1)} \ge r \\ \tau_{k(i)-1} - \tau_{k(i-1)} < r \end{cases} \Rightarrow \tau_{k(N)} \le t_0 + \tau + (r + \tau)(N-1) = t_0 + T$.

now, we prove step B by induction method.

For $\tau_{k(1)} = \tau_1$ $\quad A = \hat{a}(\rho_1) \xRightarrow{\text{by ② + } \hat{b}(\gamma) < A} m(t) \le \hat{a}(\rho_1) - d$.

$\tau_{k(2)}$ $\quad A = \hat{a}(\rho_1) - d \xrightarrow{\text{by ② + } \hat{b}(\gamma) < A} m(t) \le \hat{a}(\rho_1) - 2d$

$\Rightarrow m(t) \le \hat{a}(\rho_1) - Nd \le (\hat{b}(\gamma) + Nd) - Nd = \hat{b}(\gamma)$ $\forall t \ge t_0 + T$.

Through $\hat{b}(E(\|x(t)\|^2) \le m(t) \le \hat{b}(\gamma) \Rightarrow E(\|x(t)\|^2) \le \gamma$, $\forall t \ge T + t_0$ □

Remark 4.1.

(1) Theorem 4.1: unstable $\xRightarrow{\text{impulsive effect.}}$ stable.

(2) (iv) $M_2 > M_1$ ensure that any possible growth in V between impulsive is reduced by V at the impulsives.

(3) $M_1 \Rightarrow \tilde{M}_1 := \sup_{k \in \mathbb{N}} \int_{\tau_{k-1}}^{\tau_k} p(s)ds$

(4) For m.s. uniform stability. $\tau < \infty$ is not necessary.

(5). (iii) is independent of the time delay.

(6). p th moment stability $\quad b(\|\psi(0)\|) \le V(t, \psi(0)) \le a(\|\psi(0)\|^p)$, (i)

exponential stability $\quad a(\quad) = a \cdot \|\psi(0)\|^p \quad b(\quad) = b \cdot \|\ \| \quad c = c \cdot \|\cdot\|$

(7). For sufficient condition of stability property for system 4.1.

we need (ii) $\Rightarrow LV \le -p(t)\, c(V(t, \psi))$, $V(t+s, \psi) \le \bar{g}(V(t, \psi(0))$

(iv) $\Rightarrow M_2 = \sup_{q \ge 0} \int_q^{\bar{g}(q)} \frac{ds}{c(s)} < \inf_{t \ge 0} \int_t^{t+\mu} p(s)ds < \infty$

$\mu = \inf\{\tau_k - \tau_{k-1}\}$. More detail we refer to.

Corollary 4.1. □

(b) : Th4.3. The stabilit[illegible]

(3). 4.2 Stability Analysis. by comparison method.

Similar method in ODE.

Th4.2: $\exists\, a \in K_2$: $\liminf_{s\to\infty} a(s) > 0$. and (A1), (A2).

Let $V \in \mathcal{Y}^{1,2}([-r,\infty)\times\mathbb{R}^n : \mathbb{R}_+)$ satisfy

(i) $V(t, \varphi(0)) \le a(\|\varphi(0)\|^2) \le a(\|\varphi\|_r^2)$. $\quad$ ($q \in K_3$. (non-dreasing))

(ii) $\mathcal{L}V \le h(t, v)$ $\forall t \ne \tau_k$. when. $V(t+s, \varphi) \le q(V(t, \varphi))$.

h : continuous, $h(t, z)$ is concave in z.

$\lim_{(t,y)\to(\tau_k^-, x)} h(t, y) = h(\tau_k^-, x)$. exists.

(iii) $\forall \tau_k$, $V + \cancel{g} \le \alpha_k(V(\tau_k^-, \varphi(0^-)))$, α_k is non-dreasing concave function

(iv). the auxiliary scalar impulsive system.

$$\begin{cases} D^+ v(t) = h(t, v(t)) & t \ne \tau_k \\ v(t) = \alpha_k(v(t^-)) & t = \tau_k \\ v(t_0) = v_0 \ge 0. \end{cases}$$

has a maximal solution $r(t) = r(t, t_0, v_0)$.

Then. $E[V(t_0, x_0)] < v_0$ implies $E[V] < r(t)$. $\forall t \ge t_0$.

proof: Let $x(t)$ be any solution of system (4.1).

By Itô formula and condition (ii), we have, $\forall t \in [\tau_{k-1}, \tau_k)$

$$\underbrace{E[V(t)]}_{=: m(t)} \le E(V(\tau_{k-1})) + \int_{\tau_{k-1}}^{t} E\, h(s, V(s))\, ds$$

Jensen inequality

$$\le E(V(\tau_{k-1})) + \int_{\tau_{k-1}}^{t} h(s, E(V(s)))\, ds$$

$\Rightarrow$ $D^+ m(t) \le h(t, m(t))$.

By (iii) $\Rightarrow$ $m(\tau_k) \le \alpha_k(m(\tau_k^-))$. In summary, we have

$$\begin{cases} D^+ m(t) \le h(t, m(t)) \\ m(\tau_k) \le \alpha_k(m(\tau_k^-)) \\ m(t_0) = E(V(t_0, x_0)). \end{cases}$$

Therefore., by standard comparison method, (See Th 1.6.1 in [2])

we have $m(t) \le r(t)$ ~~[illegible]~~ ; $\forall t \ge t_0$. □

⑥ : Th 4.3. The stability properties of auxiliary Scalar system (4.6) $\Rightarrow$ stability properties of (4.1).

proof: (i) (4.6) stability $\Rightarrow$ (4.1) is stable

We have $\forall\, b(\varepsilon)>0,\ \exists\, \delta>0$. s.t $v_0<\delta \Rightarrow v(t,t_0,v_0)\le b(\varepsilon), \forall t\ge t_0$

choose $v_0 = a(\|\phi\|_r^2)$. $\delta_1=\delta_1(\varepsilon)>0$ such that $a(\delta_1)<b(\varepsilon) < \delta$.

$\hat{\delta}=\min\{\delta, \delta_1\}$. we claim that

$$E[\|\phi\|_r^2]\le \hat{\delta} \Rightarrow E[\|x(t)\|^2]<\varepsilon, \forall t\ge t_0$$

If our claim were not true., $\exists\, \bar{t}\in[\tau_k, \tau_{k+1})$.

$$E[\|x(\bar{t})\|^2]\ge\varepsilon. \qquad E[\|x(t)\|^2]<\varepsilon.\ \forall t\in[\tau_k, \bar{t}).$$

By (A1) $\Rightarrow \exists\, \underline{t}$, st $\tau_k<\underline{t}\le\bar{t}$ satisfying $\underline{\varepsilon\le E(\|x(\underline{t})\|^2)<\rho}$.

Define $m(t)=E(V(t))$. $\forall t\in[t_0, \underline{t}]$. By Th 4.2, we get.

$$m(t)<r(t;t_0, a(E(\|\phi\|_r^2))), \forall t\in[t_0,\underline{t}]$$

By condition (i), we obtain $\underbrace{\ }_{b\text{ convex}}$

$$b(\varepsilon)\le b(E(\|x(\underline{t})\|^2)\le E(b(\|x(t)\|^2)\le E(V(\underline{t}))=m(\underline{t}).$$

$$b(\varepsilon)\le m(\underline{t})\le r(t,t_0,a(E(\|\phi\|_r^2)))\le r(t,t_0,a(\hat{\delta}))<b(\varepsilon).$$

contradiction. Therefore.. we have $E[\|x(t)\|^2]<\varepsilon.\ \forall t\ge t_0$.

(ii) (4.6) uniformly stable $\Rightarrow$ (4.1) uniformly stable.

Choose $0<\eta<\rho_1<\rho$. Assume that. $\forall\, b(\eta)>0$. $\exists\,\delta>0$. $T=T(\eta,\varepsilon)>0$ st $v_0<\delta \Rightarrow$ ~~[illegible]~~ $r(t;t_0,v_0)<b(\eta)$. $\forall t\ge t_0+T$.

Similar to (i). we omit it here. $\square$.

Corollary 4.2. Let $\alpha_k(V(\tau_k^-,\psi(0^-)))=\alpha(d_k)V(\tau_k^-,\psi(0^-))$.

where $d_k\ge 0$ and $d=\sum_{k=1}^{\infty} d_k<\infty$, $\alpha(d_k)\ge 1\ \forall k$ $(\prod_{k=1}^{\infty}\alpha(d_k)<\infty)$

If (i) $h=0$. provided. that $V(t+s,\psi(s))\le q(V(t,\psi(0)))$ for some $s\in[-r,0]$. $q\in K3$, then $x\equiv0$ of (4.1) is uniform stable in the m.s.

(ii) $h(t,V(t,\psi(0))=-c\,(V(t,\psi(0)))$, q defined in (i) then. $x\equiv0$ is asymptotically stable in the m.s.

proof: Step A: V is well defined, i.e, $E[\|x(t)\|^2]\le\rho_1$, for some $t\ge0$.
Step B. construct comparison scalar system
Step C. C.1 $t^*\in[\tau_{k-1},\tau_k)$ (no impulsive) C.2. $t^*\in[\tau_k,\tau_{k+1})$, $k\ge 1$ (impulsive.

7. ~~Step A~~ Let $0<\varepsilon<\rho_1$, $1 \le \bar{d} = \prod_{k=1}^{\infty} \alpha(d_k) < \infty$. $\delta = \delta(\varepsilon) < \hat{a}^{-1}(\hat{b}(\varepsilon)/\bar{d}) \Rightarrow 0<\delta<\varepsilon$.
$b^{-1}(\delta)\le\varepsilon$ $(\hat{b}<\hat{a})$.

Let $t_0 \in [\tau_{l-1}, \tau_l)$ for some positive integer l and ϕ, for which $E[\|\phi\|_r^2] \le \delta$. We show $x(t)$ is uniformly stable, i.e. $E[\|x(t)\|] \le \varepsilon$.

• Step A:

If not, $\exists t^*$ st. $\forall t \in [t_0 - r, t^*)$, $E[\|x(t)\|^2] < \varepsilon < \rho_1$.

and either $E[\|x(t^*)\|^2] = \varepsilon$, $E[\|x(t^*)\|^2] = E(\|x_0\|_r^2) = \varepsilon$

or $E[\|x(t^*)\|^2] > \varepsilon$ $t^* = \tau_k$ for some k.

by (4.1) $\varepsilon < E[\|x(t^*)\|^2] < \rho$ since $E[\|x(t^{*-})\|^2] \le \varepsilon < \rho_1$.

Thus $V(t, x(t))$ is well-defined for $t \in [t_0, t^*]$

• Step B: Applying Itô formula to V and recall (i)

$$E(V(t)) \le E(V(s)) + E\int_s^t LV(u, x(u))\,du.$$

Define $m(t) = EV(t) \Rightarrow D^+m(t) \le E(LV(t)) \le E(LV(t)) \le 0$.

provided $m(t+s) \le q(m(t))$ (This means $m(t)$ is nonincreasing, $\forall t\in[t_0, t^*]$).

Furthermore, $\forall \tau_k \in [t_0, t^*]$, $m(\tau_k) \le \alpha(d_k) m(\tau_k^-)$.

construct comparison impulsive system $\begin{cases} D^+v(t) = 0 & t \ne \tau_k \\ v(t) = \alpha(d_k) v(t^-) & t = \tau_k \\ v(t_0) = v_0 = m_0 = E[V(t_0, x_0)]. \end{cases}$

• Step C: (C.1) $t_0 \in [\tau_{l-1}, \tau_l)$ $t^* \in [\tau_{l-1}, \tau_l)$

by $D^+v(t) \le 0 \Rightarrow E[V(t)] \le v(t) \le v_0 < \delta < \varepsilon$. $\forall t \in [t_0, t^*]$

$$b(E(\|x(t)\|^2)) \le EV(t) \le \delta < \hat{a}^{-1}\left(\frac{\hat{b}(\varepsilon)}{\bar{d}}\right).$$

$\Rightarrow E(\|x(t)\|^2) \le \varepsilon$ (since $b^{-1}(\delta) < \varepsilon$.)

C.2. $t^* \in [\tau_k, \tau_{k+1})$, $k \ge l$.

$$\overset{D^+v \le 0}{v(\tau_i) \le \alpha(d_i) v(\tau_i^-) = \alpha(d_i) v(\tau_{i-1}) \le \alpha(d_i)\alpha(d_{i-1}) v(\tau_{i-1}^-)}.$$

$$\le \prod_{i=1}^{l} \alpha(d_i) v(t_0) \le \bar{d}\, v(t_0). \quad (\alpha(d_i) > 1)$$

$$= \bar{d}\, m(t_0) \le \bar{d}\, \hat{a}(\delta). \quad (m(t_0) \le \hat{a}(\delta).)$$

with the aid of Th 4.3 we have

(definition of δ)

contradiction. $\hat{b}(\varepsilon) \le \hat{b}(E(\|x(t_0)\|^2) \le m(t^*) \le v(t^*) \le \bar{d}\,\hat{a}(\delta) < \hat{b}(\varepsilon)$. □

Remark 4.2. ① (i) and (ii) $\Rightarrow D^+E(V(t)) \le 0$

② $(4.7) \xrightarrow{\text{implusive}} V \searrow$

Remark 4.3 unstable $\xrightarrow{\text{impulsive}}$ stable, see Corollary 4.3.

⑧. Corollary 4.3. (i) $h(t,v) = p(t)\,c(V(t))$ (4.12)

(ii) $\exists \gamma_k \geq 0$. $\rho_0 > 0$. $\forall z \in (0, \rho_0)$. $\forall k \in N$.

$$\int_{T_{k-1}}^{T_k} p(s)ds + \int_z^{\alpha_k(z)} \frac{ds}{c(s)} \leq -\gamma_k \quad (4.13).$$

Then $x \equiv 0$ is uniformly stable in the m.s. If $\sum_{k=1}^{\infty} \gamma_k = \infty \Rightarrow x \equiv 0$ asymptotically stable.

proof: By Th 4.2. $m(t) = EV(t)$ leads to

$$(4.14)\begin{cases} D^+m(t) \leq p(t)\,c(m(t)) & t \neq T_k \\ m(t) \leq \alpha_k\, m(t^-) & t = T_k \\ m(t_0) = m_0 = E(V(t_0, x_0)) \end{cases} \xrightarrow[\text{system}]{\text{auxiliary}} (4.15)\begin{cases} D^+v(t) = p(t)\,c(v(t)) \\ v(t) = \alpha_k(v(t^-)) \\ v(t_0) = v_0 = m_0 \end{cases}$$

• Step A: (Stable): Let $0 < \varepsilon < \rho_0$ and $t \in [T_1, T_2)$. choose $\delta > 0$. $\delta < \min\{\varepsilon, \alpha_1(\varepsilon)\}$ and $0 < v_0 < \delta$. we claim that $v(t) < \varepsilon$. $\forall t \in [t_0, T_2)$. If not, then $\exists$ $t^* \in [t_0, T_2)$, s.t. $v(t^*) \geq \varepsilon$. Integrating (4.14)

$$\int_{\alpha_1(\delta)}^{\varepsilon} \frac{ds}{c(s)} \overset{v(t_0) < \delta < \alpha_1(\varepsilon)}{<} \int_{v(t_0)}^{v(t^*)} \frac{ds}{c(s)} \overset{(4.14)}{\leq} \int_{t_0}^{t^*} p(s)ds \overset{p \geq 0}{\leq} \int_{T_1}^{T_2} p(s)ds \quad (4.16).$$

$\Rightarrow \int_{T_1}^{T_2} p(s)\,ds + \int_{\varepsilon}^{\alpha_1(\varepsilon)} \frac{ds}{c(s)} > 0$, which contradict (4.13) $\Rightarrow v(t) < \varepsilon$. $\forall t \in [t_0, T_2)$.

Next. we prove $v(t) < \varepsilon$. $\forall t \in [T_k, T_{k+1})$ by induction method. $(V(T_k) = \alpha_k(V(T_k^-)))$

$$\int_{v(T_k^-)}^{v(t)} \frac{ds}{c(s)} = \int_{v(T_k)}^{v(t)} \frac{ds}{c(s)} + \int_{v(T_k^-)}^{\alpha_k(v(T_k^-))} \frac{ds}{c(s)} \overset{(4.16)}{\leq} \int_{T_k}^{T_{k+1}} p(s)ds + \int_{v(T_k^-)}^{\alpha_k(v_k^-)} \frac{ds}{c(s)} \overset{(4.13)}{\leq} -\gamma_k \quad (4.18)$$

$\overset{c \geq 0}{\Rightarrow} v(t) \leq V(T_k^-) \Rightarrow V(t) < \varepsilon$. $\forall t \in [T_k, T_{k+1})$. $\Rightarrow V \equiv 0$ is uniformly stable.

• Step B. (asy. stable): By (4.18) $\Rightarrow \lim_{k\to\infty} V(T_k)$ exists. We claim $\lim_{k\to\infty} V(T_k) = 0$.

If not. we assume $\lim_{k\to\infty} V(T_k) = \eta$. $\Rightarrow \frac{V(T_{k+1}) - V(T_k)}{c(\eta)} \leq \int_{V(T_k)}^{V(T_{k+1})} \frac{ds}{c(s)} \leq -\gamma_{k+1}$

where $\frac{1}{c(\eta)} = \sup\left\{\frac{1}{c(s)} \,\middle|\, \forall s \in [V(T_k), V(T_{k+1})]\right\} > 0$. $\left(\lim_{k\to\infty} V(T_k) > \eta\right)$

$\Rightarrow V(T_k) \leq V(T_{k-1}) - c(\eta)\sum_{i=1}^{k} \gamma_i \to 0 \quad \left(\frac{1}{c(\eta)} > 0 \text{ \& } \sum_{i=1}^{k} \gamma_i \to \infty\right)$.

$\Rightarrow \lim_{k\to\infty} V(T_k) = 0$. □.

Conclusion of Chap. 4.

(1). Lyapunov - Razumikhin comparison method $\Rightarrow$ stability analysis

(2) Sect 4.1 (4.1a) stable + small implusive $\Rightarrow$ stable Th 4.1

(4.1a) unstable + small implusive frequency $\Rightarrow$ stable Cor 4.1. cor. 4.3.

Sect 4.2. stability + unbounded impulsive $\Rightarrow$ stable Cor 4.2